

THE $O(1)$ -KEPLER PROBLEMS

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ABSTRACT. Let $n \geq 2$ be an integer. To each irreducible representation σ of $O(1)$, an $O(1)$ -Kepler problem in dimension n is constructed and analyzed. This system is super integrable and when $n = 2$ it is equivalent to a generalized MICZ-Kepler problem in dimension two. The dynamical symmetry group of this system is $\widetilde{Sp}_{2n}(\mathbb{R})$ with the Hilbert space of bound states $\mathcal{H}(\sigma)$ being the unitary highest weight representation of $\widetilde{Sp}_{2n}(\mathbb{R})$ with highest weight

$$(\underbrace{-1/2, \dots, -1/2}_{n-1}, -(1/2 + |\sigma|)),$$

which occurs at the right-most nontrivial reduction point in the Enright-Howe-Wallach classification diagram for the unitary highest weight modules. (Here $|\sigma| = 0$ or 1 depending on whether σ is trivial or not.) Furthermore, it is shown that the correspondence $\sigma \leftrightarrow \mathcal{H}(\sigma)$ is the theta-correspondence for dual pair $(O(1), Sp_{2n}(\mathbb{R})) \subseteq Sp_{2n}(\mathbb{R})$.

1. INTRODUCTION

The Kepler problem is a well-known physics problem in dimension three about two bodies which attract each other by a force proportional to the inverse square of their distance. What is less known about the Kepler problem is the fact that it is *super integrable*¹ at both the classical and the quantum level, and belongs to a big family of super integrable models.

One interesting such family is the family of *generalized MICZ-Kepler problems*, i.e., the family of MICZ-Kepler problems [1, 2] and their high dimensional analogues [3]. The detailed dynamical symmetry analysis of this family of super integrable models has recently been carried out in Refs. [4, 5]. It is fair to say that the study of this family of super integrable models has enriched both the field of super integrable systems and the theory of unitary highest weight modules for real non-compact Lie groups.

The purpose here is to construct and analyze yet another family of super integrable models of the Kepler type. Recall that, in the construction of generalized MICZ-Kepler problems in dimension D , the canonical bundle

$$Spin(D-1) \rightarrow Spin(D) \rightarrow S^{D-1}$$

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¹A physics model is called *super integrable* if the number of independent symmetry generators is bigger than the number of degree of freedom. For the Kepler problem, the degree of freedom is 3 and the number of independent symmetry generators is 5.

plays a pivotal role. Here the corresponding bundle is

$$\mathcal{O}(1) \rightarrow S^{n-1} \rightarrow \mathbb{RP}^{n-1}.$$

Later we shall demonstrate that, in dimension two, a model constructed in this paper is equivalent to a generalized MICZ-Kepler problem. That is why the models constructed here are called *$\mathcal{O}(1)$ -Kepler problems*.

Before stating our main result, let us fix some notations:

- n — an integer which is at least 2;
- σ — an irreducible representation of $\mathcal{O}(1)$;
- $|\sigma|$ — an integer equal to 0 (resp. 1) if σ is trivial (resp. nontrivial);
- $\tilde{\mathcal{U}}(n)$ — the nontrivial double cover of $\mathcal{U}(n)$;
- $\widetilde{\mathcal{Sp}}_{2n}(\mathbb{R})$ — the nontrivial double cover of $\mathcal{Sp}_{2n}(\mathbb{R})$.

Note that there is a natural chain of groups: $\mathrm{SO}(n) \subset \tilde{\mathcal{U}}(n) \subset \widetilde{\mathcal{Sp}}_{2n}(\mathbb{R})$.

We are now ready to state the main results on $\mathcal{O}(1)$ -Kepler problems.

Main Theorem 1. *Let $n \geq 2$ be an integer, σ an irreducible representation of $\mathcal{O}(1)$, and $|\sigma| = 0$ or 1 depending on whether σ is trivial or not.*

For the n -dimensional $\mathcal{O}(1)$ -Kepler problem with magnetic charge σ , the following statements are true:

- 1) *The bound state energy spectrum is*

$$E_I = -\frac{1/2}{(I + \frac{n}{4} + \frac{|\sigma|}{2})^2}$$

where $I = 0, 1, 2, \dots$

- 2) *There is a natural unitary action of $\widetilde{\mathcal{Sp}}_{2n}(\mathbb{R})$ on the Hilbert space $\mathcal{H}(\sigma)$ of negative-energy states, which extends the manifest unitary action of $\mathrm{SO}(n)$. In fact, $\mathcal{H}(\sigma)$ is the unitary highest weight module of $\widetilde{\mathcal{Sp}}_{2n}(\mathbb{R})$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -(\frac{1}{2} + |\sigma|))$.*

- 3) *When restricted to the maximal compact subgroup $\tilde{\mathcal{U}}(n)$, the above action yields the following orthogonal decomposition of $\mathcal{H}(\sigma)$:*

$$\mathcal{H}(\sigma) = \bigoplus_{I=0}^{\infty} \mathcal{H}_I(\sigma)$$

where $\mathcal{H}_I(\sigma)$ is a model for the irreducible $\tilde{\mathcal{U}}(n)$ -representation with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -(\frac{1}{2} + |\sigma| + 2I))$.

- 4) *$\mathcal{H}_I(\sigma)$ in part 3) is the energy eigenspace with eigenvalue E_I in part 1).*

- 5) *The correspondence between σ and $\mathcal{H}(\sigma)$ is just the theta-correspondence² for dual pair $(\mathcal{O}(1), \mathcal{Sp}_{2n}(\mathbb{R})) \subseteq \mathcal{Sp}_{2n}(\mathbb{R})$.*

For readers who are familiar with the Enright-Howe-Wallach classification diagram [6] for the unitary highest weight modules, we would like to point out that, the unitary highest weight module identified in part 2) of this theorem occurs at the rightmost nontrivial reduction point of the classification diagram, and its K -type formula is *multiplicity free* in view of part 3) of this theorem.

In section 2, we introduce the models and show that when $n = 2$ they are equivalent to the generalized MICZ-Kepler problems in dimension two. In section 3, we give a detailed analysis of the models and finish the proof of main theorem 1.

²See Ref. [7] for details on reductive dual pairs and theta-correspondence.

Depending on the interests of the readers, one may view this paper and its sequels either as a journey to discover new super integrable systems or as an effort to better understand the geometry of those Wallach representations which occur at the rightmost nontrivial reduction point of the classification diagram.

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2. THE MODELS

Let $n \geq 2$ be an integer, $\mathbb{R}_*^n = \mathbb{R}^n \setminus \{0\}$, and σ an irreducible unitary representation of $O(1)$. Consider the principal bundle

$$O(1) \rightarrow \mathbb{R}_*^n \rightarrow \widetilde{\mathbb{R}P^n}$$

where $\widetilde{\mathbb{R}P^n}$ is the quotient space of \mathbb{R}_*^n under the equivalence relation $x \sim -x$. In terms of the polar coordinates (ρ, Θ) , the Riemannian metric on $\widetilde{\mathbb{R}P^n}$ is $d\rho^2 + \rho^2 d\Theta^2$ where $d\Theta^2$ is the standard round metric on $\mathbb{R}P^{n-1}$.

Let γ_σ be the associated vector bundle attached to representation σ . It is clear that γ_σ is a flat hermitian line bundle over $\widetilde{\mathbb{R}P^n}$.

Definition 2.1. Let $n \geq 2$ be an integer and σ an irreducible representation of $O(1)$. The $O(1)$ -Kepler problem in dimension n with magnetic charge σ is the quantum mechanical system for which the wave functions are smooth sections of γ_σ , and the hamiltonian is

$$H = -\frac{1}{8\rho} \Delta \frac{1}{\rho} - \frac{1}{\rho^2}$$

where Δ is the Laplace operator twisted by γ_σ and $\rho([x]) = |x|$.

Observe that, in dimension two, \mathbb{R}_*^2 and $\widetilde{\mathbb{R}P^2}$ are diffeomorphic. We use (r, ϕ) to denote the polar coordinates on \mathbb{R}_*^2 and (ρ, θ) to denote the polar coordinates on $\widetilde{\mathbb{R}P^2}$. Let $\pi: \widetilde{\mathbb{R}P^2} \rightarrow \mathbb{R}_*^2$ be the diffeomorphism such that $\pi(\rho, \theta) = (\rho^2, 2\theta)$, then

$$\pi^*(dr^2 + r^2 d\phi^2) = 4\rho^2(d\rho^2 + \rho^2 d\theta^2) \quad \text{and} \quad \pi^*(\text{vol}_{\mathbb{R}^2_*}) = 4\rho^2 \text{vol}_{\widetilde{\mathbb{R}P^2}}.$$

Let $\mu = 0$ or $1/2$, and $\sigma_\mu: O(1) = \mathbb{Z}_2 \rightarrow \mathbb{C}$ be the group homomorphism which maps the generator of $O(1)$ to $(-1)^{2\mu}$. Let $\gamma(\mu)$ be the pullback of γ_{σ_μ} by π^{-1} . It is clear that $\gamma(\mu)$ is a flat hermitian line bundle over \mathbb{R}_*^2 .

Recall from the appendix of Ref. [8] that *the generalized MICZ-Kepler problem in dimension two with magnetic charge μ* is the quantum mechanical system for which the wave functions are smooth sections of $\gamma(\mu)$, and the hamiltonian is

$$\hat{h} = -\frac{1}{2}\Delta - \frac{1}{r}$$

where Δ is the Laplace operator twisted by $\gamma(\mu)$ and $r(x) = |x|$. We are now ready to state the following

Proposition 2.2. *The generalized MICZ-Kepler problem in dimension two with magnetic charge μ is equivalent to the $O(1)$ -Kepler problem in dimension two with magnetic charge σ_μ .*

Proof. Let Ψ_i ($i = 1$ or 2) be a wave-section for the generalized MICZ-Kepler problem in dimension two with magnetic charge μ , and

$$\psi_i(\rho, \theta) := 2\rho\pi^*(\Psi_i)(\rho, \theta) = 2\rho\Psi_i(\rho^2, 2\theta).$$

Then it is not hard to see that

$$\int_{\widetilde{\mathbb{R}P^2}} \overline{\psi_1} \psi_2 \text{vol}_{\widetilde{\mathbb{R}P^2}} = \int_{\widetilde{\mathbb{R}P^2}} \overline{\pi^*(\Psi_1)} \pi^*(\Psi_2) \pi^*(\text{vol}_{\mathbb{R}_*^2}) = \int_{\mathbb{R}_*^2} \overline{\Psi_1} \Psi_2 \text{vol}_{\mathbb{R}_*^2}$$

and

$$\begin{aligned} \int_{\widetilde{\mathbb{R}P^2}} \overline{\psi_1} H \psi_2 \text{vol}_{\widetilde{\mathbb{R}P^2}} &= \int_{\widetilde{\mathbb{R}P^2}} \overline{\pi^*(\Psi_1)} \frac{1}{\rho} H \rho \pi^*(\Psi_2) \pi^*(\text{vol}_{\mathbb{R}_*^2}) \\ &= \int_{\mathbb{R}_*^2} \overline{\Psi_1} \hat{h} \Psi_2 \text{vol}_{\mathbb{R}_*^2}. \end{aligned}$$

Here we have used the fact that

$$\begin{aligned} \frac{1}{\rho} H \rho &= -\frac{1}{8\rho^2} \left(\frac{1}{\rho} \partial_\rho \rho \partial_\rho + \frac{1}{\rho^2} \partial_\theta^2 \right) - \frac{1}{\rho^2} \\ &= -\frac{1}{2} \left(\frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\phi^2 \right) - \frac{1}{r} = \hat{h}. \end{aligned}$$

□

We end this section with an alternative definition for the O(1)-Kepler problems.

Definition 2.3 (Alternative Definition). Let n, σ be as in definition 2.1, and $|\sigma| = 0$ or 1 depending on whether σ is trivial or not. The O(1)-Kepler problem in dimension n with magnetic charge σ is the quantum mechanical system for which the wave functions are smooth complex-valued functions ψ on \mathbb{R}_*^n satisfying condition $\psi(-x) = (-1)^{|\sigma|} \psi(x)$, and the hamiltonian is

$$(2.1) \quad H = -\frac{1}{8r} \Delta \frac{1}{r} - \frac{1}{r^2}$$

where Δ is the Laplace operator on \mathbb{R}_*^n and $r(x) = |x|$.

3. THE DYNAMICAL SYMMETRY ANALYSIS

We use definition 2.3. Let ψ be the eigenfunction of H in Eq. (2.1) with eigenvalue E , so ψ is square integrable with respect to the Lebesgue measure $d\mu$, $\psi(-x) = (-1)^{|\sigma|} \psi(x)$, and

$$(3.1) \quad \left(-\frac{1}{8r} \Delta \frac{1}{r} - \frac{1}{r^2} \right) \psi = E\psi.$$

We shall solve this eigenvalue problem by separating the angles from the radius. The branching rule for $(\text{SO}(n), \text{SO}(n-1))$ plus the Fubini Reciprocity law together imply that, as modules of $\text{SO}(n)$,

$$L^2(\text{S}^{n-1}) = \bigoplus_{l=0}^{\infty} \hat{\mathcal{R}}_l$$

where \mathcal{R}_l is the irreducible and unitary representation of $\text{SO}(n)$ with the highest weight $(l, 0, \dots, 0)$.

Let $\{Y_{\mathbf{l}\mathbf{m}} \mid \mathbf{m} \in \mathcal{I}(l)\}$ be a minimal spanning set for \mathcal{R}_l . Write $\psi(x) = \tilde{R}_{kl}(r) Y_{\mathbf{l}\mathbf{m}}(\Omega)$ where $Y_{\mathbf{l}\mathbf{m}}(\Omega) \in \mathcal{R}_l$. Note that condition $\psi(-x) = (-1)^{|\sigma|} \psi(x)$ is

equivalent to equation $l \equiv |\sigma| \pmod{2}$. After separating out the angular variables, Eq. (3.1) becomes

$$(3.2) \quad \left(\frac{1}{8} \left(-\frac{1}{r^n} \partial_r r^{n-1} \partial_r \frac{1}{r} + \frac{l^2 + (n-2)l}{r^4} \right) - \frac{1}{r^2} \right) \tilde{R}_{kl} = E \tilde{R}_{kl}.$$

where $\tilde{R}_{kl} \in L^2(\mathbb{R}_+, r^{n-1} dr)$. Let $R_{k\frac{l}{2}}(t) = \tilde{R}_{kl}(\sqrt{t})/\sqrt{t}$, then we have $R_{k\frac{l}{2}} \in L^2(\mathbb{R}_+, t^{\frac{n}{2}} dt)$ and

$$(3.3) \quad \left(-\frac{1}{2t^{\frac{n}{2}}} \partial_t t^{\frac{n}{2}} \partial_t + \frac{(\frac{l}{2})^2 + (\frac{n}{2}-1)\frac{l}{2}}{2t^2} - \frac{1}{t} \right) R_{k\frac{l}{2}} = E R_{k\frac{l}{2}}.$$

By quoting results from appendix A, we have

$$(3.4) \quad E_{kl} = -\frac{1/2}{(k + \frac{l}{2} + \frac{n}{4} - 1)^2}$$

where $k = 1, 2, 3, \dots$. Let $I = k - 1 + \frac{l-|\sigma|}{2}$, then the *bound energy spectrum* is

$$(3.5) \quad E_I = -\frac{1/2}{(I + \frac{n}{4} + \frac{|\sigma|}{2})^2}$$

where $I = 0, 1, 2, \dots$; since $\tilde{R}_{kl}(r) = r R_{k\frac{l}{2}}(r^2)$, we have

$$\tilde{R}_{kl}(r) = c(k, l/2) r^{l+1} L_{k-1}^{l+\frac{n}{2}-1} \left(\frac{2}{I + \frac{n}{4} + \frac{|\sigma|}{2}} r^2 \right) \exp \left(-\frac{r^2}{I + \frac{n}{4} + \frac{|\sigma|}{2}} \right).$$

This proves part 1) of the main theorem.

For each integer $I \geq 0$, we let $\mathcal{H}_I(\sigma)$ be the linear span of

$$\{\tilde{R}_{kl} Y_{l\mathbf{m}} \mid l \equiv |\sigma| \pmod{2}, \mathbf{m} \in \mathcal{I}(l), k - 1 + \frac{l-|\sigma|}{2} = I\},$$

then

$$\mathcal{H}_I(\sigma) \cong \bigoplus_{k=0}^I \mathcal{B}_{2k+|\sigma|}$$

is the eigenspace of H with eigenvalue E_I , and the Hilbert space of bound states admits the following orthogonal decomposition into the eigenspace of H :

$$\mathcal{H}(\sigma) = \bigoplus_{I=0}^{\infty} \mathcal{H}_I(\sigma).$$

Part 4) of the main theorem is then clear. We shall show that $\mathcal{H}_I(\sigma)$ is the irreducible representation of $\tilde{U}(n)$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2} + |\sigma| + 2I)$ and $\mathcal{H}(\sigma)$ is the unitary highest weight representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2} + |\sigma|)$. To do that, we need to twist the Hilbert space of bound states and the eigenspaces.

3.1. Twisting. Let $n_I = I + \frac{n}{4} + \frac{|\sigma|}{2}$ for each integer $I \geq 0$. For each $\psi_I \in \mathcal{H}_I$, as in Refs. [9, 4], we define its twist $\tilde{\psi}_I$ by the following formula:

$$(3.6) \quad \tilde{\psi}_I(x) = c_I \frac{1}{|x|} \psi_I \left(\sqrt{\frac{n_I}{2}} x \right)$$

where $c_I > 0$ is the unique constant such that

$$(3.7) \quad \int |\tilde{\psi}_I|^2 = \int |\psi_I|^2.$$

Since

$$\left(-\frac{1}{8r}\Delta\frac{1}{r} - \frac{1}{r^2}\right)\psi_I(x) = E_I\psi_I(x),$$

after re-scaling: $x \rightarrow \sqrt{\frac{n_I}{2}}x$, we have

$$\left(-\frac{(2/n_I)^2}{8r}\Delta\frac{1}{r} - \frac{2/n_I}{r^2}\right)\psi_I(\sqrt{\frac{n_I}{2}}x) = E_I\psi_I(\sqrt{\frac{n_I}{2}}x),$$

or

$$\left(-\frac{1}{2}\Delta - 2n_I\right)\tilde{\psi}_I(x) = n_I^2 E_I r^2 \tilde{\psi}_I(x) = -\frac{1}{2}r^2 \tilde{\psi}_I(x).$$

Then

$$(3.8) \quad \begin{aligned} \left(-\frac{1}{2}\Delta + \frac{1}{2}r^2\right)\tilde{\psi}_I &= 2n_I\tilde{\psi}_I \\ &= (2I + |\sigma| + \frac{n}{2})\tilde{\psi}_I. \end{aligned}$$

We use $\tilde{\mathcal{H}}_I(\sigma)$ to denote the span of all such $\tilde{\psi}_I$'s, $\tilde{\mathcal{H}}(\sigma)$ to denote the Hilbert space direct sum of $\tilde{\mathcal{H}}_I(\sigma)$. We write the linear map sending ψ_I to $\tilde{\psi}_I$ as $\tau: \mathcal{H}(\sigma) \rightarrow \tilde{\mathcal{H}}(\sigma)$. In view of Eqs. (3.7) and (3.8), it is easy to see that τ is an linear isometry; and one can also check easily that $\tilde{\mathcal{H}}_I(\sigma)$ is the $(2I + |\sigma|)$ -th energy eigenspace of the n -dimensional isotropic harmonic isolator with hamiltonian $-\frac{1}{2}\Delta + \frac{1}{2}r^2$.

By quoting results from appendix B on the isotropic harmonic oscillators, we know that $\tilde{\mathcal{H}}_I(\sigma)$ is a model for the irreducible $\tilde{\mathbf{U}}(n)$ -representation with highest weight

$$\left(-\frac{1}{2}, \dots, -\frac{1}{2}, -(\frac{1}{2} + |\sigma| + 2I)\right),$$

and $\tilde{\mathcal{H}}(\sigma)$ is the unitary highest weight module of $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -(\frac{1}{2} + |\sigma|))$. In view of that fact that τ is an isometry, by pulling back the action of $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$ on $\tilde{\mathcal{H}}(\sigma)$ via τ , we get the action of $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$ on $\mathcal{H}(\sigma)$. Then we have parts 2), 3) and 5) of the main theorem proved.

APPENDIX A. RADIAL SCHRÖDINGER EQUATION

Let $l \geq 0$, $m > 0$ be half integers, and $l' = l + \frac{m}{2} - 1$. We are interested in finding a nonzero $R_{kl} \in L^2(\mathbb{R}_+, r^m dr)$ satisfying the radial Schrödinger equation:

$$(A.1) \quad \left(-\frac{1}{2r^m}\partial_r r^m \partial_r + \frac{l^2 + (m-1)l}{2r^2} - \frac{1}{r}\right)R_{kl}(r) = E_{k,l}R_{kl}(r).$$

for some real number $E_{k,l}$.

Solving this differential equation by the *power series method*, we can see that, for $R_{kl}(r)$ to be square integrable with respect to measure $r^m dr$, we must have

$$E_{k,l} = -\frac{1/2}{(k+l')^2}$$

for some positive integer k . With this value for E_{kl} in mind, we plug

$$R_{kl}(r) = r^{-\frac{m}{2}} y_{kl'}(r) \exp\left(-\frac{r}{k+l'}\right)$$

into Eq. (A.1), and get

$$(A.2) \quad \left(\frac{d^2}{dr^2} - \frac{2}{k+l'} \frac{d}{dr} + \left[\frac{2}{r} - \frac{l'(l'+1)}{r^2} \right] \right) y_{kl'}(r) = 0.$$

Here $y_{kl'}(r)$ is square integrable with respect to measure $\exp\left(-\frac{2r}{k+l'}\right) dr$.

Recall that, for nonnegative integers n and k , generalized Laguerre polynomial L_n^k is defined to be

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+k}).$$

Solving Eq. (A.2) in terms of power series under the square integrability condition, we arrive at the general solution of the following form:

$$y_{kl'}(r) = c(k, l) r^{l'+1} L_{k-1}^{2l'+1} \left(\frac{2}{k+l'} r \right).$$

Here $c(k, l)$ is a constant, which can be uniquely determined by requiring $c(k, l) > 0$ and $\int_0^\infty |R_{kl}(r)|^2 r^m dr = \int_0^\infty |y_{kl'}(r)|^2 \exp\left(-\frac{2r}{k+l'}\right) dr = 1$.

APPENDIX B. ISOTROPIC HARMONIC OSCILLATORS

The purpose here is to spell out the details on the quantum isotropic harmonic oscillator in dimension n . We work in the Schrödinger picture, then the hamiltonian is

$$H = \frac{1}{2} (-\Delta + r^2).$$

Here $r = \sqrt{\sum_{i=1}^n (x^i)^2}$ and $\Delta = \sum_{i=1}^n \partial_i^2$.

Let $a_k = \frac{1}{\sqrt{2}}(x_k + \partial_k)$ and a_k^\dagger be the hermitian adjoint of a_k . a_k 's are called annihilation operators and a_k^\dagger 's are called the creation operators. It is a standard fact that all bound states are created from the ground state $|\Omega\rangle$ via the creation operators. By definition, a k -th *excited state* is a state created from the ground state via a degree k polynomial in creation operators. For example, $(a_1^\dagger - 3a_2^\dagger)|\Omega\rangle$ is a first excited state, and $(a_1^\dagger a_2^\dagger + 3(a_2^\dagger)^2)|\Omega\rangle$ is a second excited state.

Introducing operators

$$\begin{aligned} -H_i &= a_i^\dagger a_i + \frac{1}{2} & \text{for } 1 \leq i \leq n, \\ E_{-e^j + e^k} &= a_j^\dagger a_k & \text{for } 1 \leq j < k \leq n, \\ E_{-e^j - e^k} &= a_j^\dagger a_k^\dagger & \text{for } 1 \leq j < k \leq n, \\ E_{-2e^j} &= \frac{1}{\sqrt{2}} a_j^\dagger a_j^\dagger & \text{for } 1 \leq j \leq n. \end{aligned}$$

It can be verified that the Hamiltonian can be written as

$$H = -\sum_i H_i = \sum_{k=1}^n a_k^\dagger a_k + \frac{n}{2};$$

moreover, operators H_i , $E_{-e^j \pm e^k}$, $E_{-e^j \pm e^k}^\dagger$ satisfy the commutation relations of a Cartan-Chevalley basis for $\mathfrak{sp}_{2n}(\mathbb{R})$.

It is then clear that the ground state is non-degenerate and has highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2})$, the 1st excited states are degenerate and form the unitary highest weight representation of $\mathfrak{u}(n)$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -\frac{3}{2})$. In general, the I -th excited states form the unitary highest weight representation of $\mathfrak{u}(n)$ with highest weight $(-\frac{1}{2}, \dots, -\frac{1}{2}, -(\frac{1}{2} + I))$.

Under $\mathfrak{sp}_{2n}(\mathbb{R})$, the Hilbert space of bound states splits into two irreducible components: the one consisting of states with even number of particles and the one consisting of states with odd number of particles.

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